

Once again about quantum deformations of $D = 4$ Lorentz algebra: twistings of q -deformation

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Abstract

This paper together with the previous one [1] presents the detailed description of all quantum deformations of $D = 4$ Lorentz algebra as Hopf algebra in terms of complex and real generators. We describe here in detail two quantum deformations of the $D = 4$ Lorentz algebra $\mathfrak{o}(3, 1)$ obtained by twisting of the standard q -deformation $U_q(\mathfrak{o}(3, 1))$. For the first twisted q -deformation an Abelian twist depending on Cartan generators of $\mathfrak{o}(3, 1)$ is used. The second example of twisting provides a quantum deformation of Cremmer-Gervais type for the Lorentz algebra. For completeness we describe also twisting of the Lorentz algebra by standard Jordanian twist. By twist quantization techniques we obtain for these deformations new explicit formulae for the deformed coproducts and antipodes of the $\mathfrak{o}(3, 1)$ -generators.

1 Introduction

The quantization of gravity is not only important as the completion of the quantum description of fundamental interactions - it affects also the basic structure of space time and the nature of relativistic symmetries (see e.g [2]-[4]). One can conjecture that the noncommutative space-time and quantum symmetries, described by noncommutative Hopf algebra, provide an algebraic deformation of the classical symmetry framework which is caused by the quantum gravity corrections. In the center of all relativistic considerations is the Lorentz symmetry, and therefore all possible modifications of Lorentz symmetries should be carefully studied. The following two ways of studying the Lorentz symmetry deformations have been proposed:

(i) The simplest way is obtained by considering nonlinear realizations of classical Lorentz symmetries (see e.g. [5]-[7]) obtained usually by a nonlinear transformation of the four-momentum basis. In such a way we can mainly interpret the effects due to the modification of relativistic mass shell condition (see e.g. [8]). In such a framework the space-time manifold remains commutative, i.e. one can use the methods of classical geometry and classical group theory.

(ii) The quantum extension of a symmetry group is provided by the noncommutative quasitriangular Hopf-algebras, with representation spaces described by non-commutative modules. This technique was extensively applied to relativistic symmetries (see e.g. [9]-[16]).

In the present paper the second way of modifying relativistic symmetries is investigated. The formalism of quasitriangular Hopf algebras [17] describing the deformations of universal enveloping algebras and the corresponding dual quantum groups were extensively studied (see e.g. [18]-[21]) in order to describe quantum modification of physical symmetries and introduce the noncommutative geometry in physics. Very important from the point of view of possible physical applications are the quantum Hopf-algebraic deformations of the Lorentz and Poincaré algebras.

The Hopf-algebraic deformations are described infinitesimally by the Poisson structures, satisfying homogenous (standard) or inhomogeneous (modified) Yang-Baxter equations. The Poisson structures for Lorentz algebra are well-known and have been classified some time ago by S. Zakrzewski [22] (see also [23]) who provided basic four classical $\mathfrak{o}(3,1)$ r -matrices. Two $D = 4$ Lorentz r -matrices generate the Jordanian and extended Jordanian deformations of $\mathfrak{o}(3,1)$. The extended Jordanian deformation was considered in detail for complex basis as well as real basis of $\mathfrak{o}(3,1)$ [1, 24]. The Jordanian deformation is defined by the well-known Jordanian twist and Hopf structure of this deformation is presented in this paper. Remaining two r -matrices generate quantum deformations which are less known, and are studied explicitly in the present paper. We use extensively the property that these deformations can be described as twisting of q -deformed Hopf algebra $U_q(\mathfrak{o}(3,1))$ [25, 26]. It should be stressed that most of twisting procedures considered in the literature are imposed on the classical Lie algebra structures. In this paper we consider the twists modifying already quantum-deformed Hopf-algebraic symmetry¹.

The plan of our paper is the following. In Sect.2 the complete list of classical r -matrices (see [22]) is presented in terms of real and complex generators. Subsequently we describe in Sect.3 the explicit Hopf algebra structure of the Lorentz algebra quantized by Jordanian twist [27]. First twisting of the q -deformed Lorentz algebra $U_q(\mathfrak{o}(3,1))$ is obtained by using the Abelian twist in Sect.4, which is a function of the Cartan subalgebra of $U_q(\mathfrak{o}(3,1))$. In this case explicit formulae of Hopf structure are given in terms of the complex and real Cartan-Weyl basis of $U_q(\mathfrak{o}(3,1))$. We show also that the inclusion $U_q(\mathfrak{o}(3,1)) \supseteq U_q(\mathfrak{o}(3))$ can be realized on the algebra level after suitable choice of the basis for $U_q(\mathfrak{o}(3))$ ². In Sect.5 we use a q -Abelian twist and as a result we obtain the quantum deformation for the Lorentz algebra of Cremmer-Gervais type [29]-[31]. Explicit formulae of Hopf structure for this case are also given in terms of the complex and real Cartan-Weyl basis of $U_q(\mathfrak{o}(3,1))$. The Sect.6 is Outlook. In last Sect.7 we consider some specialization of q -Hadamard formula which describes a similarity transformation by a q -exponential.

¹This inclusion is different from the Hopf-algebraic one presented by Podleś and Woronowicz [9], where $U_q(\mathfrak{sl}(2))$ is constructed as the extension of $U_q(\mathfrak{su}(2))$ via double product construction.

²For an analogous twist modifying κ -deformation of the Poincaré algebra see [28].

It should be recalled that the q -deformation does not permit the extension of the deformation of Lorentz symmetry to q -deformed Poincaré algebra in the framework of standard Hopf algebra, with the coproducts defined on standard tensor products. If we wish to find the deformed counterpart of Poincaré algebra extending Drinfeld-Jimbo deformation of Lorentz algebra, we have to consider the class of braided quantum Poincaré algebras [32], which require as well the deformation of tensor categories. On the other side one can show [13, 33] that the q -deformation of the Lorentz symmetry can be embedded as a Hopf subalgebra in the q -deformed Weyl algebra obtained by adding to the Poincaré algebra the dilatation generators.

2 $D = 4$ Lorentz algebra and its classical r -matrices

Firstly we remind some information from [1]. The classical canonical basis of the $D = 4$ Lorentz algebra, $\mathfrak{o}(3, 1)$, can be described by anti-Hermitian six generators $(h, e_{\pm}, h', e'_{\pm})$ satisfying the following non-vanishing commutation relations³:

$$[h, e_{\pm}] = \pm e_{\pm}, \quad [e_+, e_-] = 2h, \quad (2.1)$$

$$[h, e'_{\pm}] = \pm e'_{\pm}, \quad [h', e_{\pm}] = \pm e'_{\pm}, \quad [e_{\pm}, e'_{\mp}] = \pm 2h', \quad (2.2)$$

$$[h', e'_{\pm}] = \mp e_{\pm}, \quad [e'_+, e'_-] = -2h, \quad (2.3)$$

and moreover

$$a^* = -a \quad (\forall a \in \mathfrak{o}(3, 1)). \quad (2.4)$$

A complete list of classical r -matrices which describe all Poisson structures and generate quantum deformations for $\mathfrak{o}(3, 1)$ involve the four independent formulas [22]:

$$r_1 = \alpha(e_+ \wedge h - e'_+ \wedge h') + 2\beta e'_+ \wedge e_+, \quad (2.5)$$

$$r_2 = \alpha e_+ \wedge h, \quad (2.6)$$

$$r_3 = \alpha(e'_+ \wedge e_- + e_+ \wedge e'_-) + \beta(e_+ \wedge e_- - e'_+ \wedge e'_-) - 2\gamma h \wedge h', \quad (2.7)$$

$$r_4 = \alpha(e'_+ \wedge e_- + e_+ \wedge e'_- - 2h \wedge h') \pm e_+ \wedge e'_+. \quad (2.8)$$

All r -matrices are skew-symmetric, i.e. $r_j^{21} = -r_j^{12}$. Moreover if the universal R -matrices R_{r_j} ($j = 1, 2, 3, 4$) of the quantum deformations corresponding to the classical r -matrices (2.5)–(2.8) are unitary then these r -matrices are anti-Hermitian, i.e.

$$r_j^* = -r_j \quad (j = 1, 2, 3, 4). \quad (2.9)$$

Therefore the $*$ -operation (2.4) should be lift to the tensor product $\mathfrak{o}(3, 1) \otimes \mathfrak{o}(3, 1)$. There are two variants of this lifting: *direct* and *flipped* [34], namely,

$$(a \otimes b)^* = a^* \otimes b^* \quad (* - \text{direct}), \quad (2.10)$$

$$(a \otimes b)^* = b^* \otimes a^* \quad (* - \text{flipped}). \quad (2.11)$$

³Since the real Lie algebra $\mathfrak{o}(3, 1)$ is standard realification of the complex Lie $\mathfrak{sl}(2, \mathbb{C})$ these relations are easy obtained from the defining relations for $\mathfrak{sl}(2, \mathbb{C})$, i.e. from (2.1).

We see that if the "direct" lifting of the $*$ -operation (2.4) is used then all parameters in (2.5)–(2.8) are pure imaginary. In the case of the "flipped" lifting (2.11) all parameters in (2.5)–(2.8) are real.

The first r -matrix (2.5) satisfies the homogeneous CYBE and it is Jordanian type. Corresponding quantum deformation for the case (2.10) was described detailed in the paper [1] and it is entire defined by the extended Jordanian twist⁴:

$$F_{r_1} = \exp \left(\frac{\imath\beta}{\alpha^2} \sigma \wedge \varphi \right) \exp (h \otimes \sigma - h' \otimes \varphi) , \quad (2.12)$$

$$\sigma = \frac{1}{2} \ln [(1 + \alpha e_+)^2 + (\alpha e'_+)^2] , \quad \varphi = \arctan \frac{\alpha e'_+}{1 + \alpha e_+} \quad (2.13)$$

The second r -matrix (2.6) also satisfies the homogeneous CYBE and it is the standard Jordanian r -matrix. Corresponding twist provided the condition (2.10) is presented in the next section.

The last two r -matrices (2.7) and (2.8) satisfy the non-homogeneous (modified) CYBE and they can be easily obtained from solutions of the complex algebra $\mathfrak{o}(4, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ which is complexification of $\mathfrak{o}(3, 1)$. Indeed, let us introduce the complex basis of Lorentz algebra ($\mathfrak{o}(3, 1) \simeq \mathfrak{sl}(2; \mathbb{C}) \oplus \overline{\mathfrak{sl}}(2, \mathbb{C})$) described by two commuting sets of complex generators:

$$H_1 = \frac{1}{2} (h + \imath h') , \quad E_{1\pm} = \frac{1}{2} (e_{\pm} + \imath e'_{\pm}) , \quad (2.14)$$

$$H_2 = \frac{1}{2} (h - \imath h') , \quad E_{2\pm} = \frac{1}{2} (e_{\pm} - \imath e'_{\pm}) , \quad (2.15)$$

which satisfy the relations (compare with (2.1))

$$[H_k, E_{k\pm}] = \pm E_{k\pm} , \quad [E_{k+}, E_{k-}] = 2H_k \quad (k = 1, 2) . \quad (2.16)$$

The $*$ -operation describing the real structure acts on the generators H_k , and $E_{k\pm}$ ($k = 1, 2$) as follows

$$H_1^* = -H_2 , \quad E_{1\pm}^* = -E_{2\pm} , \quad H_2^* = -H_1 , \quad E_{2\pm}^* = -E_{1\pm} . \quad (2.17)$$

The classical r -matrix r_3 and r_4 in terms of the complex basis (2.9), (2.10) take the form

$$\begin{aligned} r_3 &= r'_3 + r''_3 , \\ r'_3 &:= 2(\beta + \imath\alpha) E_{1+} \wedge E_{1-} + 2(\beta - \imath\alpha) E_{2+} \wedge E_{2-} , \\ r''_3 &:= 4\imath\gamma H_2 \wedge H_1 , \end{aligned} \quad (2.18)$$

and

$$\begin{aligned} r_4 &= r'_4 + r''_4 , \\ r'_4 &:= 2\imath\alpha (E_{1+} \wedge E_{1-} - E_{2+} \wedge E_{2-} - 2H_1 \wedge H_2) , \\ r''_4 &:= 2\imath\nu E_{1+} \wedge E_{2+} . \end{aligned} \quad (2.19)$$

For the sake of convenience we introduce parameter ν in⁵ r''_4 . It should be noted that r'_3 , r''_3 and r'_4 , r''_4 are themselves classical r -matrices. We see that the r -matrix r'_3 is simply a sum of two standard r -matrices of $\mathfrak{sl}(2; \mathbb{C})$, satisfying the anti-Hermitian condition $r^* = -r$. Analogously, it is not hard to see that the r -matrix r_4 corresponds to a Belavin-Drinfeld triple [29] for the Lie algebra $\mathfrak{sl}(2; \mathbb{C}) \oplus \overline{\mathfrak{sl}}(2, \mathbb{C})$. Indeed, applying the Cartan automorphism $E_{2\pm} \rightarrow E_{2\mp}$, $H_2 \rightarrow -H_2$ we see that this is really correct (see also [31]).

⁴This twist is in accord with the r -matrix (2.5) in the sense of a formula of the type (4.18).

⁵We can reduce this parameter ν to ± 1 by automorphism of $\mathfrak{o}(4, \mathbb{C})$.

3 Hopf structure of Jordanian deformation for Lorentz algebra $\mathfrak{o}(3, 1)$

Whereas the quantum twist F_{r_2} corresponding to the classical Jordanian r -matrix (2.7) was well known for a long time [27]

$$F_{r_2} = \exp(h \otimes \sigma), \quad \sigma = \ln(1 + \alpha e_+) , \quad (3.1)$$

however we did not find in a literature any complete Hopf structure for this Jordanian deformation of the Lorentz algebra $\mathfrak{o}(3, 1)$. In this Section we present this Hopf structure, namely, we give explicit formulas for the co-products $\Delta_{r_2}(\cdot) := F_{r_2} \Delta(\cdot) F_{r_2}^{-1}$ and antipodes $S_{r_2}(\cdot) = u S(\cdot) u^{-1}$ of the Jordanian deformation of Lorentz algebra $\mathfrak{o}(3, 1)$ for all classical canonical basis $(h, e_{\pm}, h', e'_{\pm})$. Here $\Delta(\cdot)$ and $S(\cdot)$ are primitive (non-deformed), i.e. $\Delta(a) = a \otimes 1 + 1 \otimes a$ and $S(a) = -a$ for $\forall a \in \{h, e_{\pm}, h', e'_{\pm}\}$, and u is given as follows (see [35])

$$u = m(\text{id} \otimes S)(F_{r_2}) = \exp(-\alpha h e_+) . \quad (3.2)$$

Using a twist technics presented in Section III.D of the paper [36] it is not hard to calculate the following formulas for the deformed co-products $\Delta_{r_2}(\cdot)$ for all canonical basis $(h, e_{\pm}, h', e'_{\pm})$:

$$\Delta_{r_2}(h) = h \otimes e^{-\sigma} + 1 \otimes h , \quad (3.3)$$

$$\Delta_{r_2}(h') = h' \otimes 1 + 1 \otimes h' - \alpha h \otimes e'_+ e^{-\sigma} , \quad (3.4)$$

$$\Delta_{r_2}(e_+) = e_+ \otimes e^{\sigma} + 1 \otimes e_+ , \quad (3.5)$$

$$\Delta_{r_2}(e'_+) = e'_+ \otimes e^{\sigma} + 1 \otimes e'_+ , \quad (3.6)$$

$$\Delta_{r_2}(e_-) = e_- \otimes e^{-\sigma} + 1 \otimes e_- + 2\alpha h \otimes h e^{-\sigma} - \alpha^2 h(h-1) \otimes e_+ e^{-2\sigma} , \quad (3.7)$$

$$\Delta_{r_2}(e'_-) = e'_- \otimes e^{-\sigma} + 1 \otimes e'_- + 2\alpha h \otimes h' e^{-\sigma} - \alpha^2 h(h-1) \otimes e'_+ e^{-2\sigma} . \quad (3.8)$$

It should be noted that the formulas (3.3), (3.5) and (3.7) can be found in [27] (cf. also [37] where superextension of the Jordanian twist has been described). Using the formula (3.2) one can easily calculate the formulas of the deformed antipodes $S_{r_2}(\cdot)$:

$$S_{r_2}(h) = -h e^{\sigma} , \quad S_{r_2}(h') = -h' - \alpha h e'_+ , \quad (3.9)$$

$$S_{r_2}(e_+) = -e_+ e^{-\sigma} , \quad S_{r_2}(e'_+) = -e'_+ e^{-\sigma} , \quad (3.10)$$

$$S_{r_2}(e_-) = -e_- e^{\sigma} + 2\alpha h^2 e^{\sigma} + \alpha^2 h(h-1) e_+ e^{\sigma} , \quad (3.11)$$

$$S_{r_2}(e'_-) = -e'_- e^{\sigma} + 2\alpha h h' e^{\sigma} + \alpha^2 h(h-1) e'_+ e^{\sigma} . \quad (3.12)$$

4 Twisted q -deformation of Cartan type for Lorentz algebra $\mathfrak{o}(3, 1)$

In this Section we explicitly describe quantum deformation corresponding to the classical r -matrix r_3 (2.13). Since the r -matrix r_3'' is Abelian and it co-commutes with r_3' (see [34, 35])

therefore we firstly quantize $\mathfrak{o}(3, 1)$ in the direction r'_3 and then we apply an Abelian twist corresponding to the r -matrix r''_3 .

For the sake of convenience we introduce the following notations $z_{\pm} := \beta \pm i\alpha$. It should be noted that $z_- = z_+^*$ if the parameters α and β are real, and $z_- = -z_+^*$ if the parameters α and β are pure imaginary. From structure of the classical r -matrix r'_3 , (2.19), follows that a quantum deformation $U_{r'_3}(\mathfrak{o}(3, 1))$ is a combination of two q -analogs of $U(\mathfrak{sl}(2; \mathbb{C}))$ with the parameter q_{z_+} and q_{z_-} , where $q_{z_{\pm}} := \exp z_{\pm}$. Thus $U_{r'_3}(\mathfrak{o}(3, 1)) \cong U_{q_{z_+}}(\mathfrak{sl}(2; \mathbb{C})) \otimes U_{q_{z_-}}(\mathfrak{sl}(2; \mathbb{C}))$ and the standard generators $q_{z_+}^{\pm H_1}$, $E_{1\pm}$ and $q_{z_-}^{\pm H_2}$, $E_{2\pm}$ satisfy the following non-vanishing defining relations

$$q_{z_+}^{H_1} E_{1\pm} = q_{z_+}^{\pm 1} E_{1\pm} q_{z_+}^{H_1}, \quad [E_{1+}, E_{1-}] = \frac{q_{z_+}^{2H_1} - q_{z_+}^{-2H_1}}{q_{z_+} - q_{z_+}^{-1}}, \quad (4.1)$$

$$q_{z_-}^{H_2} E_{2\pm} = q_{z_-}^{\pm 1} E_{2\pm} q_{z_-}^{H_2}, \quad [E_{2+}, E_{2-}] = \frac{q_{z_-}^{2H_2} - q_{z_-}^{-2H_2}}{q_{z_-} - q_{z_-}^{-1}}. \quad (4.2)$$

In this case the co-product $\Delta_{r'_3}$ and antipode $S_{r'_3}$ for can be given by the formulas:

$$\Delta_{r'_3}(q_{z_+}^{\pm H_1}) = q_{z_+}^{\pm H_1} \otimes q_{z_+}^{\pm H_1}, \quad \Delta_{r'_3}(E_{1\pm}) = E_{1\pm} \otimes q_{z_+}^{H_1} + q_{z_+}^{-H_1} \otimes E_{1\pm}, \quad (4.3)$$

$$\Delta_{r'_3}(q_{z_-}^{\pm H_2}) = q_{z_-}^{\pm H_2} \otimes q_{z_-}^{\pm H_2}, \quad \Delta_{r'_3}(E_{2\pm}) = E_{2\pm} \otimes q_{z_-}^{H_2} + q_{z_-}^{-H_2} \otimes E_{2\pm}, \quad (4.4)$$

$$S_{r'_3}(q_{z_+}^{\pm H_1}) = q_{z_+}^{\mp H_1}, \quad S_{r'_3}(E_{1\pm}) = -q_{z_+}^{\pm 1} E_{1\pm}, \quad (4.5)$$

$$S_{r'_3}(q_{z_-}^{\pm H_2}) = q_{z_-}^{\mp H_2}, \quad S_{r'_3}(E_{2\pm}) = -q_{z_-}^{\mp 1} E_{2\pm}. \quad (4.6)$$

The $*$ -involution describing the real structure on the generators (2.14) and (2.15) can be adapted to the quantum generators $q_z^{\pm H_1}$, $E_{1\pm}$ and $q_z^{\pm H_2}$, $E_{2\pm}$ as follows

$$(q_{z_+}^{\pm H_1})^* = q_{z_+}^{\mp H_1}, \quad E_{1\pm}^* = -E_{2\pm}, \quad (q_{z_-}^{\pm H_2})^* = q_{z_-}^{\mp H_2}, \quad E_{2\pm}^* = -E_{1\pm}, \quad (4.7)$$

and there exit two $*$ -liftings: *flip* and *direct*, namely,

$$(a \otimes b)^* = a^* \otimes b^* \quad (* - \text{direct}), \quad (4.8)$$

$$(a \otimes b)^* = b^* \otimes a^* \quad (* - \text{flipped}) \quad (4.9)$$

for any $a \otimes b \in U_{r'_3}(\mathfrak{o}(3, 1)) \otimes U_{r'_3}(\mathfrak{o}(3, 1))$, where $*$ -direct involution corresponds to the case of the pure imaginary parameters α , β and $*$ -flipped involution corresponds to the case of the real deformation parameters α , β . It should be stressed that the Hopf structure on $U_{r'_3}(\mathfrak{o}(3, 1))$, (4.3)–(4.6), satisfy the consistency conditions under the $*$ -involution

$$\Delta_{r'_3}(a^*) = (\Delta_{r'_3}(a))^*, \quad S_{r'_3}((S_{r'_3}(a^*))^*) = a \quad (\forall a \in U_{r'_3}(\mathfrak{o}(3, 1))). \quad (4.10)$$

The universal R -matrix, $R_{r'_3}$, which connects the direct $\Delta_{r'_3}^{12} := \Delta_{r'_3}$ and opposite $\Delta_{r'_3}^{21}$ co-products

$$R_{r'_3} \Delta_{r'_3}^{12}(a) = \Delta_{r'_3}^{21}(a) R_{r'_3} \quad (\forall a \in U(\mathfrak{o}(3, 1))) \quad (4.11)$$

has the form:

$$R_{r'_3} = R'_1 R'_2 = R'_2 R'_1, \quad (4.12)$$

where

$$R'_1 = \exp_{q_{z_+}^{-2}} \left((q_{z_+} - q_{z_+}^{-1}) E_{1+} q_{z_+}^{-H_1} \otimes q_{z_+}^{H_1} E_{1-} \right) q_{z_+}^{2H_1 \otimes H_1}, \quad (4.13)$$

and

$$R'_2 = \exp_{q_{z_-}^{-2}} \left((q_{z_-} - q_{z_-}^{-1}) E_{2+} q_{z_-}^{-H_2} \otimes q_{z_-}^{H_2} E_{2-} \right) q_{z_-}^{2H_2 \otimes H_2} \quad (4.14)$$

for *-direct involution, i.e. when the parameters α, β are pure imaginary, and

$$R'_2 = \exp_{q_{z_-}^2} \left((q_{z_-}^{-1} - q_{z_-}) E_{2-} q_{z_-}^{-H_2} \otimes q_{z_-}^{H_2} E_{2+} \right) q_{z_-}^{-2H_2 \otimes H_2} \quad (4.15)$$

for *-flipped involution, i.e. when the parameters α, β are real. Here we use the standard definition of q -exponential

$$\exp_q(x) := \sum_{n \geq 0} \frac{x^n}{(n)_q!}, \quad (n)_q! := (1)_q (2)_q \cdots (n)_q, \quad (n)_q = \frac{1 - q^n}{1 - q}. \quad (4.16)$$

The universal R -matrix (4.10) is unitary

$$R_{r'_3}^* = R_{r'_3}^{-1}. \quad (4.17)$$

In the limit $z \rightarrow 0$ we have

$$R_{r'_3} = 1 + r_{BD} + O(z^2), \quad (4.18)$$

where r_{BD} is the classical Belavin-Drinfeld r -matrix

$$r_{BD} = 2z_+ (E_{1+} \otimes E_{1-} + H_1 \otimes H_1) + 2z_- (E_{2+} \otimes E_{2-} + H_2 \otimes H_2) \quad (4.19)$$

for *-direct involution, i.e. when the parameters α, β are pure imaginary, and

$$r_{BD} = 2z_+ (E_{1+} \otimes E_{1-} + H_1 \otimes H_1) - 2z_- (E_{2-} \otimes E_{2+} + H_2 \otimes H_2) \quad (4.20)$$

for *-flipped involution, i.e. when the parameters α, β are real. These r -matrix are not skew-symmetric they satisfy the condition

$$r_{BD}^{12} + r_{BD}^{21} = \Omega \quad (4.21)$$

where Ω is the split anti-Hermitian Casimir element of $\mathfrak{o}(3, 1)$ ⁶

$$\begin{aligned} \Omega &= 2z_+ (E_{1+} \otimes E_{1-} + E_{1-} \otimes E_{1+} + 2H_1 \otimes H_1) \\ &\quad - 2z_+^* (E_{2+} \otimes E_{2-} + E_{2-} \otimes E_{2+} + 2H_2 \otimes H_2) \\ &= \imath(z_+ + z_+^*)(e_+ \otimes e'_- + e'_- \otimes e_+ + e'_+ \otimes e_- + e_- \otimes e'_+ + h \otimes h' + h' \otimes h) \\ &\quad + (z_+ - z_+^*)(e_+ \otimes e_- + e_- \otimes e_+ - e'_+ \otimes e'_- - e'_- \otimes e'_+ + h \otimes h - h' \otimes h'). \end{aligned} \quad (4.22)$$

⁶Here in (4.22) and also in (4.19), (4.20) the generators $E_{1\pm}, E_{2\pm}$, and e_{\pm}, e'_{\pm} are not deformed.

The Belavin-Drinfeld r -matrix r_{BD} satisfies the homogeneous classical Yang-Baxter equation and the r -matrix r'_3 is a skew-symmetric part of r_{BD} , namely

$$r_{BD} = \frac{1}{2} r'_3 + \frac{1}{2} \Omega . \quad (4.23)$$

Now we consider deformation of the quantum algebra $U_{r'_3}(\mathfrak{o}(3,1))$ (secondary quantization of $U(\mathfrak{o}(3,1))$) corresponding to the additional r -matrix r''_3 , (2.18). Since the generators H_1 and H_2 have the trivial coproduct

$$\Delta_{r'_3}(H_k) = H_k \otimes 1 + 1 \otimes H_k \quad (k = 1, 2) , \quad (4.24)$$

therefore the unitary two-tensor

$$F_{r''_3} := q_{i\gamma}^{H_1 \wedge H_2} \quad (F_{r''_3}^* = F_{r''_3}^{-1}) \quad (4.25)$$

satisfies the cocycle condition (see [38])

$$F^{12}(\Delta_{r'_3} \otimes \text{id})(F) = F^{23}(\text{id} \otimes \Delta_{r'_3})(F) , \quad (4.26)$$

and the "unital" normalization condition

$$(\epsilon \otimes \text{id})(F) = (\text{id} \otimes \epsilon)(F) = 1 , \quad (4.27)$$

where ϵ is a counit. Thus the complete deformation corresponding to the r -matrix r_3 is the twisted deformation of $U_{r'_3}(\mathfrak{o}(3,1))$, i.e. the resulting coproduct Δ_{r_3} is given as follows

$$\Delta_{r_3}(a) = F_{r''_3} \Delta_{r'_3}(a) F_{r''_3}^{-1} \quad (\forall a \in U_{r'_3}(\mathfrak{o}(3,1))) , \quad (4.28)$$

and in this case the resulting antipode S_{r_3} does not change, $S_{r_3} = S_{r'_3}$. Applying the twisting two-tensor (4.25) to the formulas (4.3) and (4.4) we obtain

$$\Delta_{r_3}(q_{z+}^{\pm H_1}) = q_{z+}^{\pm H_1} \otimes q_{z+}^{\pm H_1}, \quad \Delta_{r_3}(q_{z-}^{\pm H_2}) = q_{z-}^{\pm H_2} \otimes q_{z-}^{\pm H_2}, \quad (4.29)$$

$$\Delta_{r_3}(E_{1\pm}) = E_{1\pm} \otimes q_{z+}^{H_1} q_{i\gamma}^{\pm H_2} + q_{z+}^{-H_1} q_{i\gamma}^{\mp H_2} \otimes E_{1\pm} , \quad (4.30)$$

$$\Delta_{r_3}(E_{2\pm}) = E_{2\pm} \otimes q_{z-}^{H_2} q_{i\gamma}^{\mp H_1} + q_{z-}^{-H_2} q_{i\gamma}^{\pm H_1} \otimes E_{2\pm} . \quad (4.31)$$

The universal R -matrix, R_{r_3} , corresponding to the resulting r -matrix r_3 , has the form

$$R_{r_3} = q_{i\gamma}^{H_2 \wedge H_1} R_{r'_1} q_{i\gamma}^{H_2 \wedge H_1} = R_1 R_2 q_{i\gamma}^{2H_2 \wedge H_1} = R_2 R_1 q_{i\gamma}^{2H_2 \wedge H_1} , \quad (4.32)$$

where

$$R_1 = \exp_{q_{z+}^{-2}} \left((q_{z+} - q_{z+}^{-1}) E_{1+} q_{z+}^{-H_1} q_{i\gamma}^{-H_2} \otimes q_{z+}^{H_1} q_{i\gamma}^{-H_2} E_{1-} \right) q_{z+}^{2H_1 \otimes H_1} , \quad (4.33)$$

and

$$R_2 = \exp_{q_{z-}^{-2}} \left((q_{z-} - q_{z-}^{-1}) E_{2+} q_{z-}^{-H_2} q_{i\gamma}^{H_1} \otimes q_{z-}^{-H_2} q_{i\gamma}^{H_1} E_{2-} \right) q_{z-}^{2H_2 \otimes H_2} \quad (4.34)$$

for $*$ -direct involution, i.e. when the parameters α, β are pure imaginary, and

$$R_2 = \exp_{q_{z_-}^2} \left((q_{z_-}^{-1} - q_{z_-}) E_{2-} q_{z_-}^{H_2} q_{i\gamma}^{H_1} \otimes q_{z_-}^{-H_2} q_{i\gamma}^{H_1} E_{2+} \right) q_{z_-}^{-2H_2 \otimes H_2} \quad (4.35)$$

for $*$ -flipped involution, i.e. when the parameters α, β are real. It is evident that the universal R -matrix (4.32) is also unitary

$$R_{r_3}^* = R_{r_3}^{-1} . \quad (4.36)$$

In the limit $z \rightarrow 0, \gamma \rightarrow 0$ we have (cf. (4.12), (4.18))

$$R_{r_3} = 1 + \tilde{r}_{BD} + O(z^2, z\gamma, \gamma^2) , \quad (4.37)$$

where

$$\tilde{r}_{BD} = \frac{1}{2} r_3 + \frac{1}{2} \Omega . \quad (4.38)$$

Now we introduce a deformed analog of the real canonical basis in the quantum algebra $U_{r_3}(\mathfrak{o}(3, 1))$ by formulas similar to the non-deformed case (2.14) and (2.15), namely

$$h = H_1 + H_2 , \quad e_{\pm} = \sqrt[4]{\lambda \lambda^{*-1}} E_{1\pm} + \left(\sqrt[4]{\lambda \lambda^{*-1}} \right)^* E_{2\pm} , \quad (4.39)$$

$$h' = i(H_2 - H_1) , \quad e'_{\pm} = i \left(\sqrt[4]{\lambda \lambda^{*-1}} \right)^* E_{2\pm} - i \sqrt[4]{\lambda \lambda^{*-1}} E_{1\pm} . \quad (4.40)$$

where entering the root factor $\sqrt[4]{\lambda \lambda^{*-1}}$ is a matter of convenience, and $\lambda := q_{z_+} - q_{z_+}^{-1}$. These basis elements are anti-Hermitian, $a^* = -a$ ($a \in \{h, h', e_{\pm}, e'_{\pm}\}$). It is evident that the commutation relations between the elements h, h' and e_{\pm}, e'_{\pm} are not deformed, that is

$$[h, e_{\pm}] = [e'_{\pm}, h'] = e_{\pm} , \quad [h, e'_{\pm}] = [h', e_{\pm}] = e'_{\pm} . \quad (4.41)$$

Using (4.1) and (4.2) we obtain the following commutation relations between the elements e_+, e'_+, e_-, e'_- :

$$[e_+, e'_+] = [e_-, e'_-] = 0 , \quad (4.42)$$

$$[e_+, e_-] = [e'_-, e'_+] = \frac{2 \sinh(xh + yh') \cosh(iyh + ixh')}{\sqrt{\cosh 2x - \cosh 2iy}} , \quad (4.43)$$

$$[e'_+, e_-] = [e_+, e'_-] = \frac{-2i \sinh(iyh + ixh') \cosh(xh + yh')}{\sqrt{\cosh 2x - \cosh 2iy}} , \quad (4.44)$$

where $x = \operatorname{Re} z_+, y = \operatorname{Im} z_+$, i.e. $x = \beta, y = \alpha$ if β, α are real and $x = -i\alpha, y = -i\beta$ if β, α are pure imaginary. It should be noted that the morphism $\omega(h) = -h, \omega(h') = -h', \omega(e_{\pm}) = -e_{\mp}, \omega(e'_{\pm}) = -e'_{\mp}$ is automorphism, i.e. ω is the Cartan automorphism. Using the expressions (4.39), (4.40), (4.29)–(4.31) and (4.3), (4.4) we obtain the following formulas of the coproducts:

$$\Delta_{r_3}(h) = h \otimes 1 + 1 \otimes h , \quad \Delta_{r_3}(h') = h' \otimes 1 + 1 \otimes h' , \quad (4.45)$$

$$\begin{aligned}\Delta_{r_3}(e_{\pm}) &= \frac{1}{2} \left(e_{\pm} \otimes (q_{z_+}^{H_1} q_{i\gamma}^{\pm H_2} + q_{z_-}^{H_2} q_{i\gamma}^{\mp H_1}) + (q_{z_+}^{-H_1} q_{i\gamma}^{\mp H_2} + q_{z_-}^{-H_2} q_{i\gamma}^{\pm H_1}) \otimes e_{\pm} \right) + \\ &+ \frac{i}{2} \left(e'_{\pm} \otimes (q_{z_+}^{H_1} q_{i\gamma}^{\pm H_2} - q_{z_-}^{H_2} q_{i\gamma}^{\mp H_1}) + (q_{z_+}^{-H_1} q_{i\gamma}^{\mp H_2} - q_{z_-}^{-H_2} q_{i\gamma}^{\pm H_1}) \otimes e'_{\pm} \right),\end{aligned}\quad (4.46)$$

$$\begin{aligned}\Delta_{r_3}(e'_{\pm}) &= \frac{1}{2} \left(e'_{\pm} \otimes (q_{z_+}^{H_1} q_{i\gamma}^{\pm H_2} + q_{z_-}^{H_2} q_{i\gamma}^{\mp H_1}) + (q_{z_+}^{-H_1} q_{i\gamma}^{\mp H_2} + q_{z_-}^{-H_2} q_{i\gamma}^{\pm H_1}) \otimes e'_{\pm} \right) + \\ &- \frac{i}{2} \left(e_{\pm} \otimes (q_{z_+}^{H_1} q_{i\gamma}^{\pm H_2} - q_{z_-}^{H_2} q_{i\gamma}^{\mp H_1}) + (q_{z_+}^{-H_1} q_{i\gamma}^{\mp H_2} - q_{z_-}^{-H_2} q_{i\gamma}^{\pm H_1}) \otimes e_{\pm} \right),\end{aligned}\quad (4.47)$$

where $H_1 = \frac{1}{2}(h + i h')$, $H_2 = \frac{1}{2}(h - i h')$. The antipodes are given as follows

$$S_{r_3}(h) = -h, \quad S_{r_3}(e_{\pm}) = -\frac{1}{2}(q_{z_+}^{\pm 1} + q_{z_+}^{\pm 1})e_{\pm} \mp \frac{i}{2}(q_{z_+}^{\pm 1} - q_{z_+}^{\pm 1})e'_{\pm}, \quad (4.48)$$

$$S_{r_3}(h') = -h', \quad S_{r_4}(e'_{\pm}) = -\frac{1}{2}(q_{z_+}^{\pm 1} + q_{z_+}^{\pm 1})e'_{\pm} \pm \frac{i}{2}(q_{z_+}^{\pm 1} - q_{z_+}^{\pm 1})e_{\pm}. \quad (4.49)$$

Turning back to the relation (4.43) it is natural to ask whether there exist another real basis vectors \tilde{e}_{\pm} , \tilde{e}'_{\pm} in which the right side of (4.43) would be a function of only Cartan element h . The answer is positive if and only if $y = 0$. This basis is given as follows

$$\tilde{h} = h, \quad \tilde{e}_{\pm} = \frac{1}{2}(e_{\pm} + i e'_{\pm})q_x^{H_2} + \frac{1}{2}(e_{\pm} - i e'_{\pm})q_x^{-H_1}, \quad (4.50)$$

$$\tilde{h}' = h', \quad \tilde{e}'_{\pm} = \frac{1}{2}(e'_{\pm} + i e_{\pm})q_x^{-H_1} + \frac{1}{2}(e'_{\pm} - i e_{\pm})q_x^{H_2}, \quad (4.51)$$

and it has the commutation relations

$$[\tilde{h}, \tilde{e}_{\pm}] = [\tilde{e}'_{\pm}, \tilde{h}'] = \pm \tilde{e}_{\pm}, \quad [\tilde{h}, \tilde{e}'_{\pm}] = [\tilde{h}', \tilde{e}_{\pm}] = \pm \tilde{e}'_{\pm}, \quad (4.52)$$

$$[\tilde{e}_+, \tilde{e}_-] = [\tilde{e}'_-, \tilde{e}'_+] = \frac{\sinh 2x \tilde{h}}{\sinh x}, \quad [\tilde{e}_{\pm}, \tilde{e}'_{\pm}] = \pm i \tanh \frac{x}{2} (\tilde{e}_{\pm}^2 + \tilde{e}'_{\pm}{}^2), \quad (4.53)$$

$$[\tilde{e}'_+, \tilde{e}_-] = [\tilde{e}_+, \tilde{e}'_-] = i \frac{\exp(-2ix \tilde{h}') - \cosh 2x \tilde{h}}{\sinh x}. \quad (4.54)$$

We see that the commutation relations of the elements \tilde{h} and \tilde{e}_{\pm} are the quantum q -analog of the relations (2.1), therefore the quantum algebra $U_q(\mathfrak{so}(3))$ generated by these elements is the subalgebra of $U_q(\mathfrak{o}(3,1))$, $U_q(\mathfrak{so}(3)) \subset U_q(\mathfrak{o}(3,1))$, for a real deformation parameter q , $q \in \mathbb{R}$. It should be noted that the morphism $\omega(\tilde{h}) = -h$, $\omega(\tilde{h}') = -h'$, $\omega(\tilde{e}_{\pm}) = -\tilde{e}_{\mp}$, $\omega(\tilde{e}'_{\pm}) = -\tilde{e}'_{\mp}$ is not automorphism because it does not retain the relation (4.54). Using the formulas (4.52), (4.53) and (4.46), (4.47) is not difficult to obtained formulas for the coproduct of the tilled generators. We will not write down these formulas, however, it is important to note that the subalgebra $U_q(\mathfrak{so}(3))$ is not any Hopf subalgebra of $U_q(\mathfrak{o}(3,1))$, i.e. $\Delta_{r_3}(U_q(\mathfrak{so}(3)))$ does not belong to $U_q(\mathfrak{so}(3)) \otimes U_q(\mathfrak{so}(3))$.

5 Twisted q -deformation of Lorentz algebra $\mathfrak{o}(3,1)$, corresponding to Belavin-Drinfeld triple

Next, we describe quantum deformation corresponding to the classical r -matrix r_4 (2.19). Since the r -matrix $r'_4(\alpha) := r'_4$ is a particular case of $r_3(\alpha, \beta, \gamma) := r_3$, namely $r'_4(\alpha) = r_3(\alpha, \beta =$

$0, \gamma = \alpha$), therefore a quantum deformation corresponding to the r -matrix r'_4 is obtained from the previous case by setting $\beta = 0, \gamma = \alpha$. Thus, the quantum deformation $U_{r'_4}(\mathfrak{o}(3, 1))$ is generated by the elements $q_\xi^{\pm H_1}, E_{1\pm}$ and $q_\xi^{\pm H_2}, E_{2\pm}$ with the following defining relations

$$q_\xi^{H_k} E_{k\pm} = q_\xi^{\pm 1} E_{k\pm} q_\xi^{H_k}, \quad [E_{k+}, E_{k-}] = \frac{q_\xi^{2H_k} - q_\xi^{-2H_k}}{q_\xi - q_\xi^{-1}} \quad (k = 1, 2). \quad (5.1)$$

Here and elsewhere we will set $\xi := i\alpha$, $\eta := -2i\nu$ and the parameters ξ and η can be simultaneously either real or pure imaginary. The co-product $\Delta_{r'_4}$ and antipode $S_{r'_4}$ are given by the formulas:

$$\Delta_{r'_4}(q_\xi^{\pm H_k}) = q_\xi^{\pm H_k} \otimes q_\xi^{\pm H_k} \quad (k = 1, 2), \quad (5.2)$$

$$\Delta_{r'_4}(E_{1\pm}) = E_{1\pm} \otimes q_\xi^{H_1 \pm H_2} + q_\xi^{-H_1 \mp H_2} \otimes E_{1\pm}, \quad (5.3)$$

$$\Delta_{r'_4}(E_{2\pm}) = E_{2\pm} \otimes q_\xi^{\mp H_1 - H_2} + q_\xi^{\pm H_1 + H_2} \otimes E_{2\pm}, \quad (5.4)$$

$$S_{r'_4}(q_\xi^{\pm H_k}) = q_\xi^{\mp H_k} \quad (k = 1, 2), \quad (5.5)$$

$$S_{r'_4}(E_{1\pm}) = -q_\xi^{\pm 1} E_{1\pm}, \quad S_{r'_4}(E_{2\pm}) = -q_\xi^{\mp 1} E_{2\pm}. \quad (5.6)$$

Now we want to construct deformation of the quantum algebra $U_{r'_4}(\mathfrak{o}(3, 1))$ (secondary quantization of $U(\mathfrak{o}(3, 1))$) corresponding to the additional r -matrix r''_4 , (2.19). We describe here only the pure imaginary case when the parameters ξ and η are pure imaginary, i.e. this case corresponds to the $*$ -flipped involution (4.9)⁷. Consider the two-tensor

$$F_{r''_4} := \exp_{q_\xi^2}(\eta E_{1+} q_\xi^{H_1+H_2} \otimes q_\xi^{H_1+H_2} E_{2+}). \quad (5.7)$$

It is easy to see that this two-tensor is unitary, $F_{r''_4}^* = F_{r''_4}^{-1}$, with respect to the $*$ -flipped involution. Moreover, using properties of q -exponentials (see [39]) is not hard to verify that $F_{r''_4}$ satisfies the cocycle equation (4.26). Thus the quantization corresponding to the r -matrix r_4 is the twisted q -deformation $U_{r'_4}(\mathfrak{o}(3, 1))$.

Explicit formulas of the co-products $\Delta_{r_4}(\cdot) = F_{r''_4} \Delta_{r'_4}(\cdot) F_{r''_4}^{-1}$ in the complex Cartan-Weyl bases of $U_{r'_4}(\mathfrak{o}(3, 1))$ are given as follows (see Appendix)

$$\Delta_{r_4}(q_\xi^{\pm(H_1-H_2)}) = q_\xi^{\pm(H_1-H_2)} \otimes q_\xi^{\pm(H_1-H_2)}, \quad (5.8)$$

$$\Delta_{r_4}(q_\xi^{H_1+H_2}) = \mathbb{X}^{-1} q_\xi^{H_1+H_2} \otimes q_\xi^{H_1+H_2}, \quad (5.9)$$

$$\Delta_{r_4}(q_\xi^{-H_1-H_2}) = q_\xi^{-H_1-H_2} \otimes q_\xi^{-H_1-H_2} \mathbb{X}, \quad (5.10)$$

$$\Delta_{r_4}(E_{1+}) = E_{1+} \otimes q_\xi^{H_1+H_2} + q_\xi^{-H_1-H_2} \otimes E_{1+} \mathbb{X}, \quad (5.11)$$

$$\Delta_{r_4}(E_{2+}) = E_{2+} \otimes q_\xi^{-H_1-H_2} \mathbb{X} + q_\xi^{H_1+H_2} \otimes E_{2+}, \quad (5.12)$$

⁷The real case corresponding to the $*$ -direct involution (4.8) (when ξ and η are real) is rather complicated and it will be described in another place.

$$\begin{aligned}\Delta_{r_4}(E_{1-}) &= E_{1-} \otimes q_\xi^{H_1-H_2} + q_\xi^{H_2-H_1} \otimes E_{1-} - \\ &\quad - \frac{\eta}{q_\xi - q_\xi^{-1}} (q_\xi^{-4H_1} \otimes 1 - \mathbb{X}^{-1}) (q_\xi^{3H_1+H_2} \otimes E_{2+} q_\xi^{2H_1}) ,\end{aligned}\tag{5.13}$$

$$\begin{aligned}\Delta_{r_4}(E_{2-}) &= E_{2-} \otimes q_\xi^{H_1-H_2} + q_\xi^{H_2-H_1} \otimes E_{2-} - \\ &\quad - \frac{\eta}{q_\xi - q_\xi^{-1}} (1 \otimes q_\xi^{-4H_2} - \mathbb{X}^{-1}) (E_{1+} q_\xi^{2H_2} \otimes q_\xi^{H_1+3H_2}) ,\end{aligned}\tag{5.14}$$

where

$$\mathbb{X} := 1 + \eta(q_\xi^2 - 1)E_{1+}q_\xi^{H_1+H_2} \otimes q_\xi^{H_1+H_2}E_{2+} .\tag{5.15}$$

Explicit formulas of antipodes $S_{r_4}(\cdot) = uS_{r'_4}(\cdot)u^{-1}$ where

$$u^{-1} = m \circ (S_{r'_4} \otimes \text{id}) \exp_{q_\xi^2} (\eta E_{1+} q_\xi^{H_1+H_2} \otimes q_\xi^{H_1+H_2} E_{2+}) = \exp_{q_\xi^2} (\eta E_{1+} E_{2+}) ,\tag{5.16}$$

are given as follows

$$S_{r_4}(q_\xi^{H_1-H_2}) = q_\xi^{-(H_1-H_2)} , \quad S_{r_4}(q_\xi^{-(H_1-H_2)}) = q_\xi^{H_1-H_2} ,\tag{5.17}$$

$$S_{r_4}(q_\xi^{H_1+H_2}) = q_\xi^{-H_1-H_2} X^{-1} , \quad S_{r_4}(q_\xi^{-H_1-H_2}) = X q_\xi^{H_1+H_2} ,\tag{5.18}$$

$$S_{r_4}(E_{1+}) = -q_\xi E_{1+} , \quad S_{r_4}(E_{2+}) = -q_\xi^{-1} E_{2+} ,\tag{5.19}$$

$$S_{r_4}(E_{1-}) = -q_\xi^{-1} E_{1-} + \frac{\eta}{q_\xi^2 - 1} E_{2+} q_\xi^{-2H_1} (q_\xi^{4H_1} - X^{-1}) ,\tag{5.20}$$

$$S_{r_4}(E_{2-}) = -q_\xi E_{2-} - \frac{\eta}{q_\xi^{-2} - 1} E_{1+} q_\xi^{-2H_2} (q_\xi^{4H_2} - X^{-1}) ,\tag{5.21}$$

where

$$X := 1 + \eta(q_\xi^2 - 1)E_{1+}E_{2+} .\tag{5.22}$$

Using the expressions (4.39), (4.40), and (5.8)–(5.15) we obtain the following formulas of the coproducts in terms of the real canonical basis:

$$\Delta_{r_4}(q_\alpha^{\pm h'}) = q_\alpha^{\pm h'} \otimes q_\alpha^{\pm h'} ,\tag{5.23}$$

$$\begin{aligned}\Delta_{r_4}(q_{i\alpha}^{\mp h}) &= \left(q_{i\alpha}^{-h} \otimes q_{i\alpha}^{-h} + \right. \\ &\quad \left. + \nu \sin \alpha (e_+ \otimes e_+ + e'_+ \otimes e'_+ + ie'_+ \otimes e_+ - ie_+ \otimes e'_+) \right)^{\pm 1} ,\end{aligned}\tag{5.24}$$

$$\begin{aligned}\Delta_{r_4}(e_+) &= e_+ \otimes \cos \alpha h + \cos \alpha h \otimes e_+ - e'_+ \otimes \sin \alpha h + \sin \alpha h \otimes e'_+ + \\ &\quad + \frac{\nu \sin \alpha}{2} (e_+ \otimes q_{i\alpha}^{h-1} (e_+^2 + e'_+{}^2) + (e_+^2 + e'_+{}^2) q_{i\alpha}^{h+1} \otimes e_+) + \\ &\quad + \frac{\nu \sin \alpha}{2} (e'_+ \otimes q_{i\alpha}^{h-1} (e_+^2 + e'_+{}^2) - (e_+^2 + e'_+{}^2) q_{i\alpha}^{h+1} \otimes e'_+) ,\end{aligned}\tag{5.25}$$

$$\begin{aligned}
\Delta_{r_4}(e'_+) &= e'_+ \otimes \cos \alpha h + \cos \alpha h \otimes e'_+ + e_+ \otimes \sin \alpha h - \sin \alpha h \otimes e_+ + \\
&+ \frac{\nu \sin \alpha}{2} \left(e'_+ \otimes q_{i\alpha}^{h-1} (e_+^2 + e'_+{}^2) + (e_+^2 + e'_+{}^2) q_{i\alpha}^{h+1} \otimes e'_+ \right) - \\
&- \frac{\nu \sin \alpha}{2} \left(e_+ \otimes q_{i\alpha}^{h-1} (e_+^2 + e'_+{}^2) - (e_+^2 + e'_+{}^2) q_{i\alpha}^{h+1} \otimes e_+ \right),
\end{aligned} \tag{5.26}$$

$$\begin{aligned}
\Delta_{r_4}(e_-) &= e_- \otimes q_\alpha^{-h'} + q_\alpha^{h'} \otimes e_- + \\
&+ \frac{\nu(\sin \alpha)^{-1}}{2} \left(e_+ q_{i\alpha}^{h-ih'} \otimes q_\alpha^{-h'} + q_\alpha^{h'} \otimes q_{i\alpha}^{h+ih'} e_+ \right) + \\
&+ \frac{\nu(\sin \alpha)^{-1}}{2} \left(e'_+ q_{i\alpha}^{h-ih'} \otimes q_\alpha^{-h'} - q_\alpha^{h'} \otimes q_{i\alpha}^{h+ih'} e'_+ \right) - \\
&- \frac{\nu(\sin \alpha)^{-1}}{2} \left(e_+ q_\alpha^{h'} \otimes q_{i\alpha}^{h-ih'} \Delta_{r_4}(q_{i\alpha}^h) + \Delta_{r_4}(q_{i\alpha}^h) q_{i\alpha}^{h+ih'} \otimes q_\alpha^{-h'} e_+ \right) - \\
&- \frac{\nu(\sin \alpha)^{-1}}{2} \left(e'_+ q_\alpha^{h'} \otimes q_{i\alpha}^{h-ih'} \Delta_{r_4}(q_{i\alpha}^h) - \Delta_{r_4}(q_{i\alpha}^h) q_{i\alpha}^{h+ih'} \otimes q_\alpha^{-h'} e'_+ \right),
\end{aligned} \tag{5.27}$$

$$\begin{aligned}
\Delta_{r_4}(e'_-) &= e'_- \otimes q_\alpha^{-h'} + q_\alpha^{h'} \otimes e'_- - \\
&- \frac{\nu(\sin \alpha)^{-1}}{2} \left(e'_+ q_{i\alpha}^{h-ih'} \otimes q_\alpha^{-h'} + q_\alpha^{h'} \otimes q_{i\alpha}^{h+ih'} e'_+ \right) + \\
&+ \frac{\nu(\sin \alpha)^{-1}}{2} \left(e_+ q_{i\alpha}^{h-ih'} \otimes q_\alpha^{-h'} - q_\alpha^{h'} \otimes q_{i\alpha}^{h+ih'} e_+ \right) + \\
&+ \frac{\nu(\sin \alpha)^{-1}}{2} \left(e'_+ q_\alpha^{h'} \otimes q_{i\alpha}^{h-ih'} \Delta_{r_4}(q_{i\alpha}^h) + \Delta_{r_4}(q_{i\alpha}^h) q_{i\alpha}^{h+ih'} \otimes q_\alpha^{-h'} e'_+ \right) - \\
&- \frac{\nu(\sin \alpha)^{-1}}{2} \left(e_+ q_\alpha^{h'} \otimes q_{i\alpha}^{h-ih'} \Delta_{r_4}(q_{i\alpha}^h) - \Delta_{r_4}(q_{i\alpha}^h) q_{i\alpha}^{h+ih'} \otimes q_\alpha^{-h'} e_+ \right).
\end{aligned} \tag{5.28}$$

Here we remaind that $\xi = i\alpha$, $\eta = -2\nu$ where parameters α and ν are real. Explicit formulas of antipodes are given as follows

$$S_{r_4}(q_\alpha^{\pm h'}) = q_\alpha^{\mp h'}, \tag{5.29}$$

$$S_{r_4}(q_{i\alpha}^{\mp h}) = \left((1 - 2\nu(q_{i\alpha}^2 - 1)(e_+^2 + e'_+{}^2)) q_{i\alpha}^h \right)^{\pm 1}, \tag{5.30}$$

$$S_{r_4}(e_+) = -\cos \alpha e_+ + \sin \alpha e'_+, \tag{5.31}$$

$$S_{r_4}(e'_+) = -\cos \alpha e'_+ - \sin \alpha e_+, \tag{5.32}$$

$$\begin{aligned}
S_{r_4}(e_-) &= -\cos \alpha e_- - \sin \alpha e'_- - \\
&- \frac{\nu(\sin \alpha)^{-1}}{2} \left(e_+ q_{i\alpha}^{h+ih'-1} + q_{i\alpha}^{h-ih'+1} e_+ \right) + \\
&+ \frac{\nu(\sin \alpha)^{-1}}{2} \left(e'_+ q_{i\alpha}^{h+ih'-1} - q_{i\alpha}^{h-ih'+1} e'_+ \right) + \\
&+ \frac{\nu(\sin \alpha)^{-1}}{2} \left(e_+ q_{i\alpha}^{-ih'-1} S_{r_4}(q_{i\alpha}^h) + S_{r_4}(q_{i\alpha}^h) q_{i\alpha}^{ih'+1} e_+ \right) - \\
&- \frac{\nu(\sin \alpha)^{-1}}{2} \left(e'_+ q_{i\alpha}^{-ih'-1} S_{r_4}(q_{i\alpha}^h) - S_{r_4}(q_{i\alpha}^h) q_{i\alpha}^{ih'+1} e'_+ \right),
\end{aligned} \tag{5.33}$$

$$\begin{aligned}
S_{r_4}(e_-) = & -\cos \alpha e'_- + \sin \alpha e_- + \\
& + \frac{\nu(\sin \alpha)^{-1}}{2} \left(e'_+ q_{i\alpha}^{h+ih'-1} + q_{i\alpha}^{h-ih'+1} e'_+ \right) + \\
& + \frac{\nu(\sin \alpha)^{-1}}{2} \left(e_+ q_{i\alpha}^{h+ih'-1} - q_{i\alpha}^{h-ih'+1} e_+ \right) - \\
& - \frac{\nu(\sin \alpha)^{-1}}{2} \left(e'_+ q_{i\alpha}^{-ih'-1} S_{r_4}(q_{i\alpha}^h) + S_{r_4}(q_{i\alpha}^h) q_{i\alpha}^{ih'+1} e'_+ \right) - \\
& - \frac{\nu(\sin \alpha)^{-1}}{2} \left(e_+ q_{i\alpha}^{-ih'-1} S_{r_4}(q_{i\alpha}^h) - S_{r_4}(q_{i\alpha}^h) q_{i\alpha}^{ih'+1} e_+ \right) .
\end{aligned} \tag{5.34}$$

6 Outlook

In this paper we presented in detail the Hopf algebra structures describing three quantum deformations of $D = 4$ Lorentz algebra $\mathfrak{o}(3, 1)$. Two of them are obtained by twisting of the standard q -deformation $U_q(\mathfrak{o}(3, 1))$. For the first twisted q -deformation an Abelian twist depending on Cartan generators of $\mathfrak{o}(3, 1)$ was used. The second example of twisting provides a quantum deformation of Cremmer-Gervais type for the Lorentz algebra. By twist quantization techniques we obtained explicit formulae for the deformed coproducts and antipodes of the $\mathfrak{o}(3, 1)$ -generators. For the sake of completeness we also incorporated here non-standard Jordanian deformation, while the remaining extended Jordanian twist has been considered with details in [1, 24].

The next step in the programme of explicit description of quantum deformations of the relativistic symmetries is to look for twists of quantum deformations of the Poincaré algebra, described by modified classical r -matrices. For that purpose one should find firstly full classification of $D = 4$ Poincaré modified classical r -matrices by completing the results in [40]. It should be add that recently (see [41]) we also started the systematic study of twists describing quantum deformations of Poincaré superalgebra.

7 Appendix

Here we consider some specialization of a q -deformed Hadamard lemma⁸, which is main tool for calculations of our results in Sect.5.

Let A and B be two arbitrary elements of some quantum algebra and let $\exp_q(A)$ be a formal q -exponential (4.16) of the element A . The formal q -exponential $\exp_{q^{-1}}(-A)$ is inverse to $\exp_q(A)$, i.e. $(\exp_q(A))^{-1} = \exp_{q^{-1}}(-A)$. The q -analog of Hadamard formula is given as follows (see [39])

$$\begin{aligned}
\exp_q(A) B (\exp_q(A))^{-1} &= \exp_q(A) B \exp_{q^{-1}}(-A) \equiv (\text{Ad } \exp_q(A))(B) = \\
&= \left(\sum_{n \geq 0} \frac{1}{(n)_q!} (\text{ad}_q A)^n \right) (B) = (\exp_q(\text{ad}_q A))(B) ,
\end{aligned} \tag{7.1}$$

⁸Hadamard lemma is sometimes called Baker-Campbell-Hausdorff formula.

where one sets

$$\begin{aligned} (\text{ad}_q A)^0(B) &\equiv B, \quad (\text{ad}_q(A))^1(B) \equiv [A, B], \quad (\text{ad}_q(A))^2(B) \equiv [A, [A, B]]_q, \\ (\text{ad}_q(A))^3(B) &\equiv [A, [A, [A, B]]_q]_{q^2}, \dots, (\text{ad}_q(A))^{n+1}(B) = [A, (\text{ad}_q(A))^n(B)]_{q^n}. \end{aligned} \quad (7.2)$$

Here the q -brackets $[\cdot, \cdot]_{q'}$ means $[C, D]_{q'} = CD - q'DC$.

The point is to obtain results in a compact (finite) form. It can be achieved here due to the following facts:

- Assume additionally that $[A, B]_q = 0$. Then

$$\exp_q(A) B (\exp_q(A))^{-1} = \tilde{X} B = B X. \quad (7.3)$$

- Assume $[A, B]_{q^{-1}} = 0$. Thus

$$\exp_q(A) B (\exp_q(A))^{-1} = X^{-1} B = B \tilde{X}^{-1}. \quad (7.4)$$

where $X = 1 + (q - 1)A$, $\tilde{X} = 1 - (q^{-1} - 1)A$.

In our case we have to substitute $A \rightarrow \mathbb{A} = \eta E_{1+} q_\xi^{H_1+H_2} \otimes q_\xi^{H_1+H_2} E_{2+}$ or $A = \eta E_{1+} E_{2+}$ (cf. (5.15) or (5.22)). This, due to unitarity, implies $\tilde{X} = X^*$. Therefore, each formula (5.8)–(5.21) admits its counterpart. For example, left handed version of (5.10) reads

$$\Delta_{r_4}(q_\xi^{-H_1-H_2}) = \mathbb{X}^* q_\xi^{-H_1-H_2} \otimes q_\xi^{-H_1-H_2}. \quad (7.5)$$

Further we need

$$[q_\xi^{aH_1+bH_2} \otimes q_\xi^{cH_1+dH_2}, \mathbb{A}]_{q_\xi^{a+d}} = 0 \quad \forall a, b, c, d \in \mathbb{C}. \quad (7.6)$$

Analogical expression

$$[q_\xi^{aH_1+dH_2}, A]_{q_\xi^{a+d}} = 0 \quad (7.7)$$

is valid for $A = \eta E_{1+} E_{2+}$. Now specialization to three cases $a + d = 0, 2$ or -2 together with (7.3) and (7.4) gives rise to formulae (5.8)–(5.14) and (5.17)–(5.21).

In order to calculate (5.14), for example, one decomposes

$$q_\xi^{H_2-H_1} \otimes E_{2-} = (q_\xi^{-2H_1-2H_2} \otimes E_{2-} q_\xi^{H_1+H_2}) (q_\xi^{H_1+3H_2} \otimes q_\xi^{-H_1-H_2}) \quad (7.8)$$

The second term commutes with the twist while the first one can be treated by (7.4).

Another interesting property can be realized by applying an operator $m \circ (id \otimes S_{r'_4})$ to (5.10). Then comparison with (5.18) gives

$$m \circ (id \otimes S_{r'_4})(\mathbb{X}) = X^{-1}. \quad (7.9)$$

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